Modular Schrödinger equation and dynamical duality

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We discuss quite surprising properties of the one-parameter family of modular nonlinear Schrödinger equations [G. Auberson and P. G. Sabatier, J. Math. Phys. **35**, 4028 (1994)]. We develop a unified theoretical framework for this family. Special attention is paid to the emergent dual time evolution scenarios which, albeit running in the real time parameter of the pertinent nonlinear equation, in each considered case may be mapped among each other by means of a suitable analytic continuation-in-time procedure. This dynamical duality is characteristic for nondissipative quantum motions and their dissipative (diffusion-type processes) partners, and naturally extends to classical motions in confining and scattering potentials.

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I. MOTIVATION

An inspiration for the present paper comes from the recent publication [1] discussing effects of various scale transformations upon the free Schrödinger picture dynamics. In particular, it has been noticed that an appropriate definition of the scale covariance induces Hamiltonians which mix, where a pertinent scale exponent is a hyperbolic rotation angle, an original free quantum dynamics with its free dissipative counterpart (effectively, a suitable version of the free Brownian motion) [1]; see also [2].

The two different time evolution patterns do run with respect to the same real time label. However, the ultimate "mixing" effect of the above-mentioned scale transformations takes the form of the the Lorentz-like transformation Eq. (35), where one Hamiltonian takes the role of the regular "time" and the other labels the "space" dimension. This obvious affinity with the Euclidean space-time notion and the complex analysis methods involved (e.g., Wick rotation, imaginary time transformation, analytic continuation in time) sets both conceptual and possibly phenomenological obstacles and prospects pertaining to the existence of such dynamical patterns in nature.

Most puzzling for us is the apparently dual notion of the time label which, while *a priori* referring to the real time evolution, may as well be interpreted as a Euclidean (imaginary time) evolution. It is our aim to address this issue in its full generality, by resorting to a one-parameter family of modular Schrödinger equations, with external conservative potentials.

The perspective adopted relies on standard approaches to nonlinear dynamical systems where a sensitive dependence on a control (here, the coupling strength) parameter may arise, possibly inducing global changes of properties of solutions to the equations of motion. That is exactly the case in the present analysis. We wish to demonstrate that a duality property (realized by an imaginary time transformation) does relate solutions of the modular nonlinear Schrödinger equation for various coupling constant regimes.

In Sec. II we set a general Lagrangian and Hamiltonian framework for the subsequent discussion and indicate that effectively the dynamics can be reduced to three specific coupling value choices [3,4], each of them being *separately* discussed in the literature. We aim at a stationary action principle formulation of the modular Schrödinger dynamics and the related hydrodynamics (special cases are the familiar Bohm-type quantum hydrodynamics [5] and the hydrodynamical picture of diffusion-type processes [2]). The basic principles of the action principle method are patterned after the classical hydrodynamics treatises [6,7].

In Sec. III, in Hamiltonian dynamics terms, we give a new derivation and further generalize the original arguments of [1]. We employ the scaling properties of the Shannon entropy of a continuous probability density $\rho \doteq \psi^* \psi$. The scale covariant patterns of evolution, nondissipative with an admixture of a dissipative component, are established in external potential fields.

Section IV is devoted to a detailed analysis of the emergent time duality notion for three special values 0,1,2 of the coupling parameter. There, we pay more attention to a specific relation between confining and scattering dynamical systems. A properly addressed sign issue for the potential function appears to be vital for mathematical consistency of the formalism (this links with the theory of dynamical semigroups that underlies the dissipative dynamics scenarios). As a by-product of the discussion, we establish the Lyapunov functionals for the considered dynamical patterns of behavior, and the (Shannon) entropy production rate is singled out.

Section V addresses more specific problems that allow us to grasp the duality concept from the perspective of the theory of classical conservative systems and diffusion-type (specifically Smoluchowski) stochastic processes. A number of illustrative examples is worked out in detail, with emphasis on the time duality notion in the classically inspired (standard Hamilton-Jacobi equations) and dissipative patterns of evolution (modified Hamilton-Jacobi, e.g., Smoluchowski diffusion processes vs quantum dynamics).

II. MODULAR SCHRÖDINGER EQUATIONS

Let us consider a subclass of so-called modular Schrödinger equations [3,4] which are local and homogeneous nonlinear generalizations of the standard Schrödinger equation

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$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi + \left(\kappa\frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|}\right)\psi, \tag{1}$$

where $\psi(x,t)$ is a complex function, $|\psi| \doteq (\psi^*\psi)^{1/2}$, $V(x)$ is a real function, and the coupling parameter κ is non-negative, $\kappa \geqslant 0$. If $\kappa > 0$, the pertinent nonlinear dynamics is known to preserve the $L^2(R^n)$ norm of any initially given ψ, but not the Hilbert space scalar product (ψ, ϕ) of two different, initially given ψ and ϕ. The induced dynamics is nonunitary in $L^2(R^n)$; unitarity is restored if $\kappa = 0$. The pertinent set of solutions is a subset of $L^2(R^n)$, but not a linear subspace (no superposition principle for $\kappa \neq 0$).

For all $\kappa \geqslant 0$ the usual continuity equation holds true with the familiar quantum mechanical definition of a (probability) density current: $\partial_t\rho = -\nabla\cdot j$, where $\rho = \psi^*\psi$ and $j = (\hbar/2mi)(\psi^*\nabla\psi - \psi\nabla\psi^*)$. We consider normalized solutions only, which sets a standard form of $j \doteq \rho\cdot v$, where $v = (\hbar/2mi)[(\nabla\psi/\psi) - (\nabla\psi^*/\psi^*)] \doteq (1/m)\nabla s$ is regarded as the gradient velocity field and $\rho(x,t) = |\psi|^2(x,t)$ is the probability density on R^n.

A. Lagrangian formalism

The one-parameter family of modular equations (1), together with its complex conjugate version, derives from the local Lagrangian density $\mathcal{L}$ by means of the stationary action principle [8–10,5]. Let us consider a functional of ψ functions, their space and time derivatives, including complex conjugates: $I[\psi,\psi^*] = \int_{t_1}^{t_2} L(t)dt$, where $L(t) = \int \mathcal{L}(x,t)dx$ (we leave the integration volume unspecified, possibly infinite).

We impose the stationary action condition $\delta I[\psi,\psi^*] = 0$ for independent variations $\delta\psi$, $\delta\psi^*$ which are bound to vanish at the integration volume boundaries. By invoking elements of the usual functional calculus [5], such as, e.g., $\delta\mathcal{L}/\delta\psi \equiv \partial\mathcal{L}/\partial\psi - \Sigma_i\nabla_i[\partial\mathcal{L}/\partial(\nabla_i\psi)]$, one ends up with the Euler-Lagrange equations $\partial_t[\partial\mathcal{L}/\partial(\partial_t\psi^*)] = \delta\mathcal{L}/\delta\psi^*$ and $\partial_t[\partial\mathcal{L}/\partial(\partial_t\psi)] = \delta\mathcal{L}/\delta\psi$.

If we properly specify the Lagrangian density $\mathcal{L} \doteq \mathcal{L}_\kappa$,

$$\mathcal{L}_\kappa(x,t) = \frac{i\hbar}{2}[\psi^*(\partial_t\psi) - \psi(\partial_t\psi^*)] - \frac{\hbar^2}{2m}\nabla\psi\cdot\nabla\psi^* - V(x)\psi\psi^* + \kappa\frac{\hbar^2}{8m}\left(\frac{\nabla\psi^*}{\psi^*} + \frac{\nabla\psi}{\psi}\right)^2\psi\psi^*, \tag{2}$$

the stationary action principle yields a pair of adjoint modular equations which comprise Eq. (1) in conjunction with its complex conjugate

$$-i\hbar\partial_t\psi^* = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi^* + \left(\kappa\frac{\hbar^2}{2m}\frac{\Delta|\psi|}{|\psi|}\right)\psi^*. \tag{3}$$

We have previously introduced a current velocity field $v(x,t)$ by means of $j \doteq \rho v$. Our gradient assumption $v = (1/m)\nabla s$ follows from an implicit reference to the familiar Madelung substitution $\psi = |\psi|\exp(is/\hbar)$, where $|\psi|^2 = \rho$. Its consequent exploitation allows us to rewrite the Lagrangian density (2) as follows (we leave intact the original order of the respective entries):

$$\mathcal{L}_\kappa(x,t) = -\rho\left(\partial_t s + \frac{m}{2}(u^2 + v^2) + V(x) - \kappa\frac{m}{2}u^2\right), \tag{4}$$

where $u(x,t) \doteq (\hbar/2m)\nabla\rho/\rho$ is another velocity field (named the osmotic velocity).

Here, $\delta I[\rho,s] = 0$ gives rise to a continuity equation $\partial_t\rho = -\nabla(\rho v)$ and yields

$$\partial_t s + \frac{1}{2m}(\nabla s)^2 + V + (1-\kappa)Q = 0, \tag{5}$$

where, in view of $|\psi| = \rho^{1/2}$, the familiar notion of the de Broglie–Bohm quantum potential [5,11] naturally appears:

$$Q = Q(x,t) \doteq -\frac{\hbar^2}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = -\frac{\hbar^2}{4m}\left[\frac{\Delta\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right]. \tag{6}$$

The modular Schrödinger equation (1) takes the form

$$i\hbar\partial_t\psi = [-(\hbar^2/2m)\Delta + V]\psi - \kappa Q\psi. \tag{7}$$

B. Hamiltonian formalism

A symplectic structure can be associated with the dynamical system (1)–(3) by introducing fields π_ψ and π_{ψ^*} that are conjugate to ψ and ψ^*, respectively: $\pi_\psi = \partial\mathcal{L}/\partial(\partial_t\psi) = (i\hbar/2)\psi^*$ and $\pi_{\psi^*} = \partial\mathcal{L}/\partial(\partial_t\psi^*) = -(i\hbar/2)\psi$. The subsequent Legendre-type transformation defines the Hamiltonian density

$$\begin{aligned}\mathcal{H}_\kappa &= \pi_\psi\partial_t\psi + \pi_{\psi^*}\partial_t\psi^* - \mathcal{L}_\kappa \\ &= \frac{\hbar^2}{2m}\nabla\psi\nabla\psi^* + \left[V - \kappa\frac{\hbar^2}{8m}\left(\frac{\nabla\psi^*}{\psi^*} + \frac{\nabla\psi}{\psi}\right)^2\right]\psi\psi^* \\ &= \rho\left(\frac{m}{2}v^2 + V + (1-\kappa)\frac{m}{2}u^2\right) = \pi_s\partial_t s - \mathcal{L}_\kappa,\end{aligned} \tag{8}$$

where, this time with respect to the polar fields $\rho(x,t)$ and $s(x,t)$, we have $\pi_\rho = \partial\mathcal{L}/\partial(\partial_t\rho) = 0$ and $\pi_s = \partial\mathcal{L}/\partial(\partial_t s) = -\rho$.

The Hamiltonian reads $H_\kappa(t) = \int\mathcal{H}_\kappa(x,t)dx$. For the variational calculus it is not $\mathcal{L}$ but $L = \int\mathcal{L}\,dx$ that really matters. Therefore, it is useful to note that

$$L(t) = -\langle\partial_t s\rangle - H_\kappa(t), \tag{9}$$

where, since $\int\rho\,dx = 1$, we can introduce the mean value $\langle\partial_t s\rangle = \int\rho\partial_t s\,dx$.

Let us evaluate the mean value of the generalized Hamilton-Jacobi equation (5). By assuming a proper behavior of ρ at the integration volume boundaries [11], we readily get $\langle Q\rangle \doteq \int Q\rho\,dx = +(m/2)\langle u^2\rangle > 0$. Thence, on dynamically allowed fields $\rho(t)$ and $s(x,t)$, $L(t) \equiv 0$, i.e., $\langle\partial_t s\rangle = -H_\kappa$.

Let us consider two function(al)s $A = \int\mathcal{A}(x,t)dx$ and $B = \int\mathcal{B}(x,t)dx$, which may explicitly depend on time t. We define their Poisson brackets

$$\{A,B\} = -\frac{i}{\hbar}\int dx\left(\frac{\delta A}{\delta\psi}\frac{\delta B}{\delta\psi^*} - \frac{\delta A}{\delta\psi^*}\frac{\delta B}{\delta\psi}\right). \tag{10}$$

In particular, identifying $A \equiv \psi(x,t)$ and $B \equiv H_\kappa(t)$, we get the modular Schrödinger equation (1) in the form

$$\partial_t \psi = \{\psi, H_\kappa\} \tag{11}$$

while, by setting $A \equiv \psi^*$, an adjoint equation arises,

$$\partial_t \psi^* = \{\psi^*, H_\kappa\}. \tag{12}$$

We recall, e.g., that $\dot{\pi}_\psi = -\delta H_\kappa / \delta\psi$ while $\dot{\psi} = \delta H_\kappa / \delta\psi$.

Since the time dependence of $H_\kappa(t)$ is realized only through the canonical fields, the Hamiltonian surely is a constant of motion. Thence $\langle \partial_t s \rangle$ is as well.

The polar decomposition $\psi = \rho^{1/2} \exp(is/\hbar)$, $\psi^* = \rho^{1/2} \exp(-is/\hbar)$ preserves a symplectic structure. In the self-explanatory notation it holds that

$$\{A,B\} \doteq \{A,B\}_{\psi,\psi^*} = \{A,B\}_{\rho,s} \tag{13}$$

and thence

$$\partial_t \rho = \{\rho, H_\kappa\} = \frac{\delta H_\kappa}{\delta s} = -\frac{1}{m} \nabla (\rho \nabla s) \tag{14}$$

while

$$\partial_t s = \{s, H_\kappa\} = -\frac{\delta H_\kappa}{\delta \rho} = -\frac{1}{2m} (\nabla s)^2 - V - (1-\kappa) Q. \tag{15}$$

The result is valid for all $\kappa \geqslant 0$.

Let $G(t) = \int dx\, \mathcal{G}(x,t)$. Clearly (cf. [6]), if the time dependence of G is realized only through canonically conjugate fields $\rho(x,t)$ and $s(x,t)$ (and their derivatives), then

$$\frac{dG}{dt} = \{G, H_\kappa\}. \tag{16}$$

One should realize that a particular time-dependence pattern of $G(t)$ critically relies on the chosen parameter range for $\kappa \in R^+$.

C. Reduction to effective $\kappa=0$, 1, and 2 self-coupling regimes

In the notation of Eq. (4), the particular role of coupling parameter values $\kappa=0$, 1, or 2 is very conspicuous. The relevance of just these three $\kappa=0,1,2$ parameter values may be further enhanced. Let us recall [3] important differences between the properties of solutions of Eqs. (1) and (3), depending on whether $0<\kappa<1$, $\kappa=1$, or $\kappa>1$.

(i) In the parameter range $0<\kappa<1$ (in fact $0 \leqslant \kappa < 1$), if $\psi(x,t) = |\psi| \exp(is/\hbar)$ actually is a solution of (1), then $\psi'(x',t') = |\psi'| \exp(is'/\hbar)$, with $x'=x$, $t'=(1-\kappa)^{1/2} t$, $|\psi'|(x',t') = |\psi|(x,(1-\kappa)^{-1/2} t')$, and $s'(x',t') = (1-\kappa)^{-1/2} s(x,t)$, automatically solves the linear Schrödinger equation

$$i\hbar \partial_{t'} \psi' = \left(-\frac{\hbar^2}{2m} \Delta + \frac{1}{1-\kappa} V \right) \psi'. \tag{17}$$

The scaling transformation replaces a nonlinear problem by a linear one, albeit with a rescaled potential.

(ii) For the specific value $\kappa=1$, we encounter a formalism that derives (though actually generalizing) from the wave picture of classical Newtonian mechanics [5].

(iii) In the case of $\kappa>1$, a repetition of previous scaling steps, provided we replace $(1-\kappa)^{1/2}$ by $(\kappa-1)^{1/2}$ in the pertinent formulas, results in the following outcome: $\psi'(x',t') = |\psi'| \exp(is'/\hbar)$ is a solution of the nonlinear Schrödinger equation

$$i\hbar \partial'_t \psi' = \left(-\frac{\hbar^2}{2m} \Delta + \frac{V}{\kappa - 1} \right) \psi' + 2 \left(\frac{\hbar^2}{2m} \frac{\Delta |\psi'|}{|\psi'|} \right) \psi'. \tag{18}$$

Effectively, in the whole coupling parameter range $\kappa \in R^+$, only the cases of $\kappa=0$, 1, and 2 form a mutually exclusive family, on both mathematical and physical grounds. Since for $\kappa=0$ and 2 the above scaling transformations are trivial, while being irrelevant for the distinctively borderline case of $\kappa=1$, we can safely restore the notation of Eqs. (1) and (3).

In connection with the choice of $\kappa=2$, one more observation is of utmost importance [3] (see also [12–16]). Namely, if a complex function

$$\psi(x,t) = |\psi| \exp(is/\hbar) \tag{19}$$

is a solution of (1) with $\kappa=2$, then the real function

$$\theta_*(x,t) = |\psi| \exp(-s/\hbar) \tag{20}$$

is a solution of the generalized (forward) heat equation

$$\hbar \partial_t \theta_* = \left(\frac{\hbar^2}{2m} \Delta + V \right) \theta_* \tag{21}$$

with a diffusion coefficient $D = \hbar/2m$ and an external potential $\mathcal{V} = V/2mD$. By setting $V=0$ we would arrive at the standard heat equation $\partial_t \theta_* = D \Delta \theta_*$.

Another real function $\theta(x,t) = |\psi| \exp(+s/\hbar)$ is a solution of the time-adjoint (backward) version of Eq. (21):

$$-\hbar \partial_t \theta = \left(\frac{\hbar^2}{2m} \Delta + V \right) \theta. \tag{22}$$

Note that if we had started from Eqs. (21) and (22) with the purpose of arriving at the modular equation (1) with $\kappa=2$, the ill-posed Cauchy problem would possibly become a serious obstacle. That is in view of the occurrence of the backward parabolic equation.

This ill-posedness might be healed by invoking the theory of strongly continuous dynamical semigroups [12,13,17]. To this end we need to choose $V(x)$ to be a continuous function that is bounded from above, so that $V' = -V$ becomes bounded from below. Then Eq. (21) would acquire a "canonical" form of the forward diffusion-type equation related to the contractive semigroup operator $\exp(-\hat{H}t/\hbar)$:

$$\hbar \partial_t \theta_* = -\hat{H} \theta_* = \left(\frac{\hbar^2}{2m} \Delta - V' \right) \theta_*. \tag{23}$$

Equation (23), together with its time adjoint

$$\hbar \partial_t \theta = \hat{H} \theta = \left(-\frac{\hbar^2}{2m} \Delta + V' \right) \theta, \tag{24}$$

stand for the principal dynamical equations of so-called Euclidean quantum mechanics [13], while falling into the broader framework of the Schrödinger boundary data and

stochastic interpolation problem [12,13] (see, e.g., also [17]).

Note that, if one assumes that the external potential $V'(x)$ is a continuous function that is bounded from below, then the operator $\hat{H}=-(\hbar^2/2m)\Delta+V'$ is (essentially) self-adjoint in $L^2(R^n)$. So we have consistently defined unitary transformations $\exp(-i\hat{H}t/\hbar)$ in $L^2(R^n)$ and $i\hbar\partial_t\psi=\hat{H}\psi$, together with $-i\hbar\partial_t\psi^*=\hat{H}\psi^*$ ($\kappa=0$ case), as their local manifestations.

A careful analysis reveals [12,13] that the standard Schrödinger picture dynamics can be mapped into the contractive semigroup (generalized diffusion equation) dynamics (23) and (24), by means of an analytic continuation in time. This mapping we reproduce in the book-keeping (Wick rotation) form $(it)\to t$.

We emphasize that one should not mystify the emerging "imaginary time" transformation. The time label t pertains to fairly standard (unequivocally real) dynamics scenarios. However, the detailed patterns of temporal behavior for the two motion scenarios are very different.

Nonetheless, various (real) functionals of dynamical variables, evolving in accordance with the chosen motion rule [say, Schrödinger's $\psi(t)$ and $\psi^*(t)$], may be consistently transformed into affiliated functionals that evolve according to another, we call it *dual* [dissipative $\theta^*(t)$ and $\theta(t)$] motion rule. The reverse transformation works as well. We shall address this issue below, from varied perspectives.

III. SCALE COVARIANT PATTERNS OF EVOLUTION

Let $[\Delta l]$ stand for an arbitrary (albeit fixed for the present purpose) unit of length. For a continuous probability density ρ on R^1 we can introduce its (dimensionless) Shannon entropy functional, also named the differential entropy [2],

$$S(\rho) \doteq -\int d\left(\frac{x}{[\Delta l]}\right)([\Delta l]\rho)\ln([\Delta l]\rho) = -\int dx\,\rho\ln([\Delta l]\rho). \tag{25}$$

If $\rho(x,t)$ depends on time, then $S(\rho)=S(t)$ may evolve in time as well.

The time rate equation for $S(t)$ is devoid of any $[\Delta l]$ input and for all $\kappa\geqslant 0$ has the very same functional form

$$D\dot{S} = D\{S,H_\kappa\} = -\langle uv\rangle. \tag{26}$$

Obviously, $\langle\cdot\rangle$ stands for the mean value with respect to the probability density $\rho(x,t)$.

The above time rate formula does not depend on the specific unit of length that is present in the definition of the dimensionless Shannon entropy. However, the entropy functional itself *is* sensitive to scaling transformations.

Setting $x'=x/\beta$, where $\beta>0$ is the scale parameter, we have $1=\int\rho(x)dx=\int\beta\rho(\beta x')dx'$. Accordingly, the scale transformation $x\to x'=x/\beta$ induces a transformation of the probability density in question:

$$\rho(x)\to\beta\rho(\beta x')\doteq\rho'(x'). \tag{27}$$

Given the Shannon entropy of the density $\rho(x)$, we can compare the outcome with that for the density $\rho'(x')$:

$$\begin{aligned} S'(\rho') &\doteq -\int dx'\rho'(x')\ln\{[\Delta l]\rho'(x')\} \\ &= -\int dx\,\rho(x)\ln\{\beta[\Delta l]\rho(x)\} \\ &= S(\rho)-\ln\beta. \end{aligned} \tag{28}$$

Consequently, the $x\to x/\beta$ scaling is equivalent to the sole change of the length unit $[\Delta l]\to\beta[\Delta l]$ in the Shannon entropy definition.

Since β is a fixed scaling parameter, the time rate formula (26) is scale independent. Note that, by setting $\beta\doteq\exp\alpha$, we get $S'(\rho')=S(\rho)-\alpha$.

We observe that $u'(x')=\beta u(x)$ and the scale invariance of $D\dot{\mathcal{S}}=-\langle uv\rangle$ tells us that $v'(x')=(1/\beta)v(x)$. Since we assume $v(x,t)$ to be the gradient field, $v=(1/m)\nabla s$, we readily arrive at the corresponding scaling property of $s(x,t)$:

$$s'(x')=\frac{1}{\beta^2}s(x)\to v'(x')=\frac{1}{\beta}v(x). \tag{29}$$

We demand furthermore that the ($\kappa=1$) equations $\partial_t\rho=-\nabla(\rho v)$ and $\partial_t s+(1/2m)(\nabla s)^2+V=0$ are scale invariant, i.e., they retain their form after introducing x', $\rho'(x',t)$, and $s'(x',t)$ instead of x, $\rho(x,t)$, and $s(x,t)$, respectively. This implies an *induced* scaling property of $V(x)\to V'(x')$:

$$V'(x')=\frac{1}{\beta^2}V(x) \tag{30}$$

and thus $H_1'=(1/\beta^2)H_1$.

Since

$$Q'(x',t)=\beta^2Q(x,t), \tag{31}$$

the general evolution equations for $\kappa\neq 1$ are *not* scale invariant. Let us consider the cumulative effect of the above scaling rules upon the Hamiltonian

$$H_\kappa=\int dx\,\rho[(mv^2/2)+V+(1-\kappa)(mu^2/2)] \tag{32}$$

according to

$$H_\kappa\doteq H_\kappa(t)\to H_\kappa'(t)=\int dx'\mathcal{H}(\rho',s')(x',t). \tag{33}$$

Following the guess of [1], originally analyzed in the case of $V\equiv 0$, we take $\beta=\exp(\alpha/2)$; it follows that

$$\begin{aligned} H_\kappa'(\alpha) &= \exp(-\alpha)\int dx\,\rho\left(\frac{m}{2}v^2+V\right) \\ &\quad+\exp(+\alpha)\int dx\rho(1-\kappa)\frac{m}{2}u^2, \end{aligned} \tag{34}$$

which can be recast as the hyperbolic transformation with a hyperbolic angle α,

$$H_\kappa'(\alpha)=\cosh\alpha H_\kappa-\sinh\alpha K_\kappa. \tag{35}$$

Here, in addition to H_α, we encounter a new Hamiltonian generator K_κ:

$$K_\kappa \doteq \int dx\, \rho\left(\frac{m}{2}v^2 + V - (1-\kappa)\frac{m}{2}u^2\right). \tag{36}$$

The difference between H_κ and K_κ is encoded solely in the sign inversion of the last $(1-\kappa)(mu^2/2)$ entry.

We can as well analyze the effect of the scaling transformation upon K_κ [1]:

$$K'_\kappa(\alpha) = -\sinh \alpha H_\kappa + \cosh \alpha K_\kappa. \tag{37}$$

The hyperbolic rotations that are explicit in Eqs. (35) and (37) form a conspicuous Lorentz-type transformation (hyperbolic rotation of coordinates in Minkowski space) of the direct analog $(H_\kappa, K_\kappa, 0, 0)$ of the familiar Minkowski space vectors $(p_0, p, 0, 0)$ and/or $(ct, x, 0, 0)$.

We recall that the time label t is left untouched by the scale transformations hitherto considered. Consequently, like H_κ, the generator K_κ and the induced α family of generators $H'_\kappa(\alpha)$, $K'_\kappa(\alpha)$ are legitimate Hamiltonian generators of *diverse* time evolution scenarios, all running with respect to the same time variable t. The pertinent motions arise through the symplectic structure Eq. (13) and the generic time evolution rule (16), common for all Hamiltonians and *a priori* prescribed.

Remark 1. By retracing back the passage from the modular equation to the corresponding Lagrangian and Hamiltonian, we realize that K_κ can be associated with another $(\kappa-2)$ family $(\kappa \geqslant 0)$ of modular Schrödinger equations

$$i\hbar\partial_\tau \psi = [-(\hbar^2/2m)\Delta + V]\psi + (\kappa-2)Q\psi, \tag{38}$$

to be compared with the originally introduced κ family

$$i\hbar\partial_t \psi = [-(\hbar^2/2m)\Delta + V]\psi - \kappa Q\psi. \tag{39}$$

We shall not elaborate on the general $\kappa > 2$ parameter regime and, in view of the previously established reduction procedure (Sec. II C), we confine our further discussion to the specific cases of $\kappa = 0$, 1 or 2.

Remark 2. Let us note that $K_0 \equiv H_2$, $H_1 \equiv K_1$, and $K_2 \equiv H_0$. We have $H'_1 = \exp(-\alpha)H'_1$ and the following hyperbolic transformation properties hold true for H_0 and H_2:

$$H'_0 = \cosh \alpha H_0 - \sinh \alpha H_2 \tag{40}$$

and

$$H'_2 = -\sinh \alpha H_0 + \cosh \alpha H_2. \tag{41}$$

These transformation rules are a generalization of those presented for the $V \equiv 0$ case by Brenig [1].

Remark 3. Hamiltonians of the form

$$H_0 = (m/2)\langle v^2 + u^2\rangle + \langle V\rangle \tag{42}$$

and

$$H_2 = (m/2)\langle v^2 - u^2\rangle + \langle V\rangle \tag{43}$$

are known in the literature [18,14], and are interpreted as setting the quantum-mechanical and dissipative-dynamical frameworks, respectively. Some elementary hints in this connection can also be found in [1,15,16,2]. Coming back to our observation that a rescaling of the dimensional unit $[\Delta l] \to \exp \alpha [\Delta l]$ induces a transformation $S'(\rho') = S(\rho) - \alpha$ of the Shannon entropy $S(\rho)$ for a continuous probability density ρ, we realize that the choice of $\alpha \leqslant 0$ implies a sharpening of the resolution unit; hence there is effective growth of the Shannon entropy with the lowering of $\alpha = -|\alpha|$. Indeed, we have $S'(\rho') = S(\rho) + |\alpha|$. In this situation, the effective (rescaled) Hamiltonians H'_0 and H'_2 always have an admixture of both nondissipative and dissipative components, H_0 and H_2, respectively.

IV. DYNAMICAL DUALITY

Remembering that for general functionals $G(t)$ of ρ and s we have $\dot{G} = \{G, H_\kappa\}$, let us consider a product $\mathcal{F}(x,t) \doteq -\rho(x,t)s(x,t)$ of conjugate fields s and $\pi_s = -\rho$. The time evolution of

$$F(t) = \int dx\, \mathcal{F}(x,t) \doteq -\langle s\rangle \tag{44}$$

looks quite interesting:

$$\frac{dF}{dt} = \{F, H_\kappa\} = -\int dx \left(s(x,t)\frac{\delta H_\kappa}{\delta s} - \rho(x,t)\frac{\delta H_\kappa}{\delta \rho}\right)$$
$$= -\int dx\, \rho\left(\frac{m}{2}v^2 - V - (1-\kappa)\frac{m}{2}u^2\right). \tag{45}$$

A new Hamiltonian-type functional has emerged on the right-hand side of the dynamical identity (45). Let us denote

$$H^\pm_\kappa = \int dx\, \rho\left(\frac{m}{2}v^2 \pm V \pm (1-\kappa)\frac{m}{2}u^2\right). \tag{46}$$

We point out that, in contrast to previous scaling-induced Hamiltonians, the negative sign has been generated with respect to both terms $(m/2)\langle u^2\rangle$ and $\langle V\rangle$. The $+V \to -V$ mapping was lacking in previous discussion, it is necessary to properly implement the implicit, analytic continuation-in-time procedure.

The previous motion rule is rewritten as

$$\frac{dF}{dt} = \{F, H^+_\kappa\} = -H^-_\kappa(t), \tag{47}$$

where $H^+_\kappa = H$ is the time evolution generator Eq. (8). Note that H^+_κ is here a constant of motion, while $H^-_\kappa(t)$ is not.

After accounting for the Poisson brackets (13), we encounter a complementary relationship for the time evolution that is generated by the induced Hamiltonian H^-_κ:

$$\frac{dF}{dt} = \{F, H^-_\kappa\} = -H^+_\kappa(t). \tag{48}$$

Presently, H^+_κ is a constant of motion, while $H^-_\kappa(t)$ no longer is.

For each value of $\kappa \in R^+$, we thus arrive at dual time-evolution scenarios, generated by Hamiltonians H^+_κ and H^-_κ, respectively. The duality notion is in conformity with our previous discussion (Sec. II C) of the imaginary time transformation. These dual motions, even if started from the very same initial $t = 0$ data and allowed to continue indefinitely, do

result in disparate patterns of behavior: nondissipative and dissipative, respectively.

Nonetheless, if we admit that the generator of motion chosen *a priori* is H_κ^+, then we are interested as well in

$$\frac{d^2F}{dt^2} = -\{H_\kappa^-(t), H_\kappa^+\} = -\frac{dH_\kappa^-}{dt} = +2\int \rho v \nabla [V + (1-\kappa)Q]dx, \tag{49}$$

and, presuming that H_κ^- actually is the evolution generator, in the dual evolution formula

$$\frac{d^2F}{dt^2} = -\{H_\kappa^+(t), H_\kappa^-\} = -\frac{dH_\kappa^+}{d\tau} = -2\int \rho v \nabla [V + (1-\kappa)Q]dx. \tag{50}$$

The right-hand sides of Eqs. (49) and (50) associate the mean power transfer rates (gain, loss, or none) with the dual Hamiltonian evolutions of $F(t)$, generated by H_κ^+ and H_κ^-, respectively. This remains divorced from the fact that $H_\kappa^+(t)$ and $H_\kappa^-(\tau)$ actually are constants of the pertinent dual motions.

For clarity of our discussion we shall confine further attention to $L^2(R^n)$ solutions of (1) and (3). We assume that the external potential $V(x)$ is a continuous function that is bounded from below. If the energy operator $\hat{H} = -(\hbar^2/2m)\Delta + V$ is self-adjoint, then we have consistently defined unitary transformations $\exp(-i\hat{H}t/\hbar)$ in $L^2(R)$ so that $i\hbar\partial_t\psi = \hat{H}\psi$ holds true ($\kappa = 0$).

The case of $\kappa = 1$ we associate with two classes of external potentials $\pm V(x)$, with $+V(x)$ bounded from below. This will allow us to discriminate between the confining and scattering regimes. The borderline meaning of $\kappa = 1$ can be read from Eq. (4), where the $mu^2/2$ contributions cancel.

In conformity with our previous discussion of the generalized heat equations, related to solutions of Eq. (1) with $\kappa = 2$, given $+V(x)$ and $\hat{H}$, we pass to a pair of time-adjoint parabolic equations: $\hbar\partial_t\theta_* = -\hat{H}\theta_*$ and $\hbar\partial_t\theta = \hat{H}\theta$. Here, $\theta_*(x,t) = [\exp(-\hat{H}t/\hbar)\theta_*](x,0)$ represents a forward dynamical semigroup evolution, while $\theta(x, T-t) = \exp(+\hat{H}t/\hbar)\theta(x,T)$ stands for a backward one. Both are unambiguously defined in a finite time interval $[0,T]$, provided one has prescribed suitable end-point data [12]. The corresponding modular Schrödinger equations (plus their complex conjugate versions) read

(i) $\kappa = 0 \Rightarrow i\hbar\partial_t\psi = [-(\hbar^2/2m)\Delta + V]\psi$,
(ii) $\kappa = 1 \Rightarrow i\hbar\partial_t\psi = [-(\hbar^2/2m)\Delta \pm V - Q]\psi$,
(iii) $\kappa = 2 \Rightarrow i\hbar\partial_t\psi = [-(\hbar^2/2m)\Delta - V - 2Q]\psi$.

The associated Lagrangian densities (4) in the (ρ, s) representation and the induced dynamical rules are worth listing as well. In addition to the continuity equation $\partial_t\rho = -\nabla(\rho v)$ the following Hamilton-Jacobi type equations are valid:

(i) $\kappa = 0$, $\mathcal{L} = -\rho[\partial_t s + (m/2)(v^2+u^2) + V] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 + (V+Q) = 0$,
(ii) $\kappa = 1$, $\mathcal{L} = -\rho[\partial_t s + (m/2)v^2 \pm V] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 \pm V = 0$,
(iii) $\kappa = 2$, $\mathcal{L} = -\rho[\partial_t s + (m/2)(v^2-u^2) - V] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 - (V+Q) = 0$.

On the dynamically allowed fields $\rho(t)$ and $s(x,t)$, $L(t) \equiv 0$, i.e., $\langle \partial_t s \rangle = -H$. The respective Hamiltonians (8) do follow:

(i) $H^+ \doteq \int dx\, \rho[(m/2)v^2 + V + (m/2)u^2]$,
(ii) $H_{\rm cl}^\pm \doteq \int dx\, \rho[(m/2)v^2 \pm V]$,
(iii) $H^- \doteq \int dx\, \rho[(m/2)v^2 - V - (m/2)u^2]$.

We emphasize that, from the start, $V(x)$ is chosen to be a continuous function bounded from below. In the definition of the above Hamiltonians there is no longer a κ label and the subscript "cl" refers to the classically motivated (Hamilton-Jacobi theory) wave formalism.

The evolution equations for $F = -\langle s \rangle$ (cf. [1,16,2]) clearly define dual pairs:

$$\dot{F} = \{F, H^+\} = -\int dx\, \rho\left(\frac{m}{2}v^2 - V - \frac{m}{2}u^2\right) = -H^-(t), \tag{51}$$

$$\dot{F} = \{F, H^-\} = -\int dx\, \rho\left(\frac{m}{2}v^2 + V + \frac{m}{2}u^2\right) = -H^+(t) \tag{52}$$

and

$$\dot{F} = \{F, H_{\rm cl}^+\} = -\int dx\, \rho\left(\frac{m}{2}v^2 - V\right) = -H_{\rm cl}^-(t), \tag{53}$$

$$\dot{F} = \{F, H_{\rm cl}^-\} = -\int dx\, \rho\left(\frac{m}{2}v^2 + V\right) = -H_{\rm cl}^+(t). \tag{54}$$

It is instructive to notice that the functional $F(t)$ in Eqs. (52) and (54) may consistently play the role of a Lyapunov functional, indicating the preferred sense of time ("time arrow") in the course of the evolution process. Namely, if we take $\langle V \rangle > 0$, the right-hand side expression is negative definite. Hence $F(t)$ is a monotonically decaying function of time, which is a standard signature of a dissipation process (cf. the Helmholtz free energy and the relative entropy discussion for diffusion-type processes [16,2,19]).

In the notation of Sec. II [see Eq. (8)], we have $H^\pm = H_0^\pm$ and $H_{\rm cl}^\pm = H_1^\pm$. The corresponding (dual) time rate formulas (49) and (50) do follow.

The motion rules for $\dot{F}(t)$ can be given a more transparent form by reintroducing the constants $H^\pm$ of the respective motions. Then

$$\dot{F}(t) = -m\langle v^2 \rangle(t) + H^\pm \tag{55}$$

and

$$\dot{F}(t) = -m\langle v^2 \rangle(t) + H_{\rm cl}^\pm. \tag{56}$$

Here, the non-negative term $m\langle v^2 \rangle(t)$ should receive due attention, because of its utmost importance in the study of the Shannon entropy dynamics [16,2,19], where it represents an *entropy production* time rate. The latter is generated solely by dynamical processes that are intrinsic to the system and does not involve any energy or heat flow, in or out of the potentially dissipative system (which would need the notion of a thermal reservoir external to the system).

Since H^+ and H^- are constants of the respective motions, $F(t)-tH^{\pm}$ are quantities that are monotonically decreasing in time. This property extends to the $H^{\pm}_{\rm cl}$ generated dynamics as well.

The speed (slowing down or acceleration), with which the above decay process may occur, relies on the specific dynamical pattern of behavior of $\langle v^2\rangle$. That is quantified by $-md\langle v^2\rangle/dt$; hence

$$\ddot{F}(t)=\{\dot{F}(t),H^{\pm}\}=\pm 2\int \rho v\,\nabla\,(V+Q)dx. \tag{57}$$

For $H^{\pm}_{\rm cl}$ generated motions, we have

$$\ddot{F}(t)=\{\dot{F}(t),H^{\pm}_{\rm cl}\}=\pm 2\int \rho v\,\nabla\,V\,dx. \tag{58}$$

We point out that a major distinction between the dual dynamical rules is encoded in the right-hand sides of the above equations: the conspicuous sign inversion is worth contemplation. The related integrals have the clear meaning of a power transfer (release, absorption, or possibly none), on average, that is induced by the time evolution of the pertinent dynamical system [16,2].

Let us stress that the imaginary time transformation, even if not quite explicit in our discussion, hereby has been extended to dynamical models of purely classical provenance. The pertinent Wick rotation connects confined and scattering motions, admitted to occur in a (confining) potential $+V(x)$ that is continuous and bounded from below and in its inverted (scattering) counterpart $-V(x)$, respectively. The obvious example of an inverted harmonic oscillator [20] is worth mentioning at this point. Parabolic potential barriers [21] and repulsive $1/r^2$ potentials [22] belong to the same category.

In the above discussion, the sign inversion issue is manifested in the second time derivatives of various functionals. In view of the presence of $\pm\nabla V$, we can identify this behavior as a remnant of the standard classical (Newtonian) reasoning [12]: if the sign looks wrong with respect to the classical Newton equation (e.g., we have $+\nabla V$), we can correct this "defect" by interpreting time t as an imaginary time it (or $i\tau$ to avoid notational confusion).

V. PHYSICS-RELATED IMPLEMENTATIONS OF THE DUAL DYNAMICS PATTERNS: THE ILLUSION OF IMAGINARY TIME

A. Generalities

With the notational conventions $D=\hbar/2m$, $b(x,t)\doteq v(x,t)+u(x,t)$, while imposing suitable boundary conditions (e.g., $\rho, b\rho, v\rho$ vanishing at integration boundaries or infinity), we can write the Shannon entropy time rate of change in a number of equivalent ways [2]:

$$D\dot{S}=D\{S,H_{\kappa}\}=-\langle uv\rangle=\langle v^2\rangle-\langle bv\rangle \tag{59}$$

and

$$D\dot{S}=D\langle\nabla v\rangle=\langle\nabla b\rangle+D\langle u^2\rangle. \tag{60}$$

The non-negative entry $(1/D)\langle v^2\rangle$ is interpreted as the *entropy production* rate in the considered dynamical system [16,2,19].

We note that

$$D\langle u^2\rangle=-D\langle\nabla u\rangle=\frac{2}{m}\langle Q\rangle, \tag{61}$$

so that the mean divergence of the osmotic velocity is always negative. Here $\langle Q\rangle>0$ holds true for all finite times [2]. The value 0 can be achieved, if at all, only in the asymptotic regime $t\to\infty$.

To give the flavor of the time duality (and specifically the imaginary time transformation) connection, let us mention that the free Brownian motion can be embedded in the above scheme by setting $b\equiv 0$ and regarding D as the diffusion constant. We have $v=-u=-D\nabla\ln\rho$ and the Shannon entropy time rate takes the form of the de Bruijn identity $D\dot{S}=\langle v^2\rangle=\langle u^2\rangle$ [2].

The Shannon entropy $S(t)$ of the Brownian motion grows monotonically in time, solely due to the entropy production $(1/D)\langle v^2\rangle(t)$. The latter, in turn, is known to be a decreasing function of time (at least in the large-time asymptotic) and ultimately is bound to vanish at $t\to\infty$.

In the case of a diffusion process in a conservative potential, it is not the Shannon entropy but the Helmholtz free energy that takes the role of the Lyapunov functional and sets the time arrow [see Eq. (55) and [2,19]]. Note that $F(t)-tH^-$ is a quantity monotonically decaying in time.

For comparison, the Schrödinger picture quantum dynamics typically involves $b\neq 0$. In the special case of free motion, $\dot{S}>0$; hence $S(t)$ grows indefinitely [16]. In the large-time asymptotic $\dot{S}\to 0$, while the entropy production $(1/D)\langle v^2\rangle$ remains unaffected and never vanishes while approaching a finite positive value.

However, the would-be natural property $\dot{S}(t)>0$ is not generic for quantum motion in external potentials [23]. Nonetheless, $F(t)-tH^+$ does monotonically decrease with time $t\to\infty$, indicating the Lyapunov-functional-induced arrow of time even though this dynamics is manifestly nondissipative.

B. Harmonic oscillator and its inverted partner

Let us consider a standard classical harmonic oscillator problem, where

$$H\doteq\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2 \tag{62}$$

is an obvious constant of motion for the Newtonian system $\dot{p}=m\ddot{q}=-m\omega^2q$

$$q(t)=q_0\cos\omega t+\frac{p_0}{m\omega}\sin\omega t,$$

$$p(t)=p_0\cos\omega t-m\omega q_0\sin\omega t. \tag{63}$$

Clearly $H=p_0^2/2m+(m\omega^2/2)q_0^2$ is a positive constant.

Let us perform an analytic continuation in time, by considering the Wick rotation $t\to -it$, paralleled by the transfor-

mation of the initial momentum data $p_0 \rightarrow -ip_0$. Once inserted in the above harmonic oscillator expressions, we arrive at

$$H_{-ip_0} = -p_0^2/2m + (m\omega^2/2)q_0^2 \doteq -\bar{H} \tag{64}$$

and

$$q_{-ip_0}(-it) \doteq \bar{q}(t) = q_0 \cosh \omega t - \frac{p_0}{m\omega} \sinh \omega t \tag{65}$$

together with

$$p_{-ip_0}(-it) \doteq +i\bar{p}(t) = -ip_0 \cosh \omega t + im\omega q_0 \sinh \omega t, \tag{66}$$

which can simply be rewritten as

$$\bar{p}(t) = -p_0 \cosh \omega t + m\omega q_0 \sinh \omega t. \tag{67}$$

We observe that

$$\bar{q}(-t) = q_0 \cosh \omega t + \frac{p_0}{m\omega} \sinh \omega t \tag{68}$$

and

$$-\bar{p}(-t) = p_0 \cosh \omega t + m\omega q_0 \sinh \omega t \tag{69}$$

are the familiar inverted oscillator solutions, generated by $\bar{H}$ [20].

Indeed, the equations of motion for $\bar{q}(t)$ and $\bar{p}(t)$ directly derive from the Hamiltonian $H_{-ip_0} = -\bar{H}$ with

$$\bar{H} = \frac{\bar{p}^2}{2m} - \frac{1}{2}m\omega^2\bar{q}^2. \tag{70}$$

They give rise to the Newton equation $\dot{\bar{p}} = m\ddot{\bar{q}} = +m\omega^2\bar{q}$. However, the dynamics generated by $\bar{H}$ is related to that generated by $-\bar{H}$ by time reflection: the latter dynamics runs backward, if the former runs forward.

Remark 4. At first glance, the harmonic oscillator example may be regarded as a unique special case (linear dynamical system) to which our imaginary time transformation arguments may be applied. Fortunately, we can give a number of nonlinear models whose (Euclidean) inversion can be consistently implemented, [24]. The Euclidean connection goes beyond the confining vs scattering potential idea of ours and extends to periodic potentials as well. There are examples from instanton physics: static localized (kink) solutions of the ϕ^4 nonlinear field theory in one space dimension may be interpreted as Euclidean time solutions of the double-well potential problem; the sine-Gordon kink may be interpreted as a Euclidean time solution of a plane pendulum problem.

C. Time duality via analytic continuation in time

The above procedure gives clear hints on how to connect the dual classical wave theory evolution, associated with the previously discussed Hamiltonians $H_{\mathrm{cl}}^{\pm}$. We recall that, in addition to the continuity equation, we infer the dual Hamilton-Jacobi equations $\partial_t s + (1/2m)(\nabla s)^2 \pm V = 0$ and that $\partial_t \rho = -\nabla(\rho v)$ with $v(x,t) = (1/m)\nabla s(x,t)$ holds.

In the notational convention adopted, we define the initial data $s_0(x) = -\bar{s}_0(x)$ and introduce an imaginary time transformation

$$\begin{aligned}\psi(x,t) = \rho^{1/2}\exp(is/2mD) \rightarrow \bar{\psi}(x,t) &\doteq \psi_{-is_0}(x,-it) \\ &= \rho_{-is_0}^{1/2}(x,-it)\exp[is_{-is_0}(x,-it)/2mD] \\ &\doteq \bar{\rho}^{1/2}(x,t)\exp[-\bar{s}(x,t)/2mD].\end{aligned} \tag{71}$$

We note that $\lim_{t\downarrow 0} is_{-is_0}(x,-it) = i(-is_0)(x,0) = s_0(x)$. An analogous procedure for an analytic continuation in time has been worked out in the general context of Euclidean quantum mechanics in Ref. [13], but with no mention of its extension to the classically inspired Hamilton-Jacobi equation.

Let us denote $\bar{v} = (1/m)\nabla\bar{s}$. Accordingly, the transformations implemented by (71) replace $H_{\mathrm{cl}}^{+} = \int dx\, \rho[(m/2)v^2 + V]$ by $-\bar{H}_{\mathrm{cl}}^{-} = \int dx\, \bar{\rho}[-(m/2)\bar{v}^2 + V]$, with the very same function $V(x)$ in both expressions. Clearly

$$\partial_t \rho = -\nabla(\rho v) \rightarrow \partial_t \bar{\rho} = +\nabla(\bar{\rho}\bar{v}), \tag{72}$$

which is an obvious indication of the time-reflected (backward) evolution. Analogously,

$$\partial_t s + (1/2m)(\nabla s)^2 \pm V = 0 \rightarrow \partial_t \bar{s} - (1/2m)(\nabla\bar{s})^2 + V = 0, \tag{73}$$

where the time reflection $t \rightarrow -t$ induces the expected form of the dual Hamilton-Jacobi equation:

$$\partial_t \bar{s} + (1/2m)(\nabla\bar{s})^2 - V = 0. \tag{74}$$

The analytic continuation in time discussed above directly extends to the general pair $H^{\pm}$ of dual Hamiltonians (see Sec. IV of Ref. [13]). The description becomes even more straightforward, because in this case we connect pairs of linear partial differential equations. If $\psi(x,t)$ actually is a solution of the Schrödinger equation $i(2mD)\partial_t = \hat{H}\psi$, then

$$\psi_{-is_0}(x,-it) = \bar{\rho}^{1/2}(x,t)\exp[-\bar{s}(x,t)/2mD] \doteq \theta_*(x,t) \tag{75}$$

solves a backward diffusion-type equation

$$-(2mD)\partial_t\theta_* = \hat{H}\theta_* \tag{76}$$

while

$$\theta(x,t) = \bar{\rho}^{1/2}(x,t)\exp[+\bar{s}(x,t)/2mD] \tag{77}$$

solves the forward equation

$$(2mD)\partial_t\theta = \hat{H}\theta. \tag{78}$$

In the above one may obviously identify $D = \hbar/2m$. The whole procedure can be inverted and we can trace back a nondissipative quantum dynamics pattern which stays in affinity (duality) with a given dissipative dynamics (cf. [13]).

D. Diffusion-type processes

1. Smoluchowski process

The Hamiltonian appropriate for the description of dissipative processes (strictly speaking, diffusion-type stochastic processes) has the form

$$H^- \doteq \int dx\, \rho[(m/2)v^2 - V - (m/2)u^2] \tag{79}$$

with the potential $V(x)$ which is chosen *a priori* and continuous and bounded from below. It is the functional form of $V(x)$ that determines local characteristics of the diffusion process [2].

Once the Fokker-Planck equation is inferred

$$\partial_t \rho = D\Delta\rho - \nabla(b\rho), \tag{80}$$

where $\rho_0(x)$ stands for the initial condition, we adopt the forward drift $b=f/m\gamma$ of the process in the standard Smoluchowski form, characteristic for Brownian motion in an external force field $f(x)=-\nabla\mathcal{V}$. Here, γ is a friction (damping) parameter and, instead of $D=\hbar/2m$, we prefer to think in terms of $D=k_BT/m\gamma$, where T stands for the (equilibrium) temperature of the reservoir.

An admissible form of $\mathcal{V}\rightarrow f=-\nabla\mathcal{V}$ must be compatible with a Riccatti-type equation, provided the potential function $V(x)$ has been chosen *a priori*:

$$V(x) = m\left[\frac{1}{2}\left(\frac{f}{m\gamma}\right)^2 + D\,\nabla\left(\frac{f}{m\gamma}\right)\right]. \tag{81}$$

The Fokker-Planck equation can be rewritten as a continuity equation $\partial_t\rho=-\nabla j$ with the diffusion current j in the form

$$j = \rho v = \frac{\rho}{m\gamma}(f - k_BT\,\nabla\,\ln\rho) \doteq \frac{\rho}{m}\,\nabla s. \tag{82}$$

We recall the general definition of the current velocity $v=(1/m)\nabla s$.

Since the time-independent $s=s(x)$ is here admissible, we have actually determined

$$s = -\frac{1}{\gamma}(\mathcal{V} + k_BT\ln\rho), \tag{83}$$

whose negative mean value $F=-\langle s\rangle$ determines the Helmholtz free energy of the random motion, as follows:

$$\Psi \doteq \gamma F = U - T\mathcal{S}, \tag{84}$$

where $\mathcal{S}\doteq k_BS$ stands for the Gibbs-Shannon entropy of the continuous probability distribution, while the internal energy reads $U=\langle\mathcal{V}\rangle$. Since we assume ρ and ρVv to vanish at the integration volume boundaries, we get

$$\dot{\Psi} = -(m\gamma)\langle v^2\rangle = -k_BT(\dot{\mathcal{S}})_{\text{int}} \leqslant 0. \tag{85}$$

Clearly, the Helmholtz free energy Ψ decreases as a function of time, or remains constant.

The Shannon entropy $S(t)=-\langle\ln\rho\rangle$ typically is not a conserved quantity. We impose the boundary restrictions that $\rho, v\rho, b\rho$ vanish at spatial infinity or other integration interval borders, and consider

$$D\dot{S} = \langle v^2\rangle - \langle b\cdot v\rangle, \tag{86}$$

which is rewritten as

$$\dot{S} = (\dot{S})_{\text{int}} + (\dot{S})_{\text{ext}}, \tag{87}$$

where

$$k_BT(\dot{S})_{\text{int}} \doteq m\gamma\langle v^2\rangle \geqslant 0 \tag{88}$$

stands for the entropy production rate, while

$$k_BT(\dot{S})_{\text{ext}} = -\int fj\,dx = -m\gamma\langle bv\rangle \tag{89}$$

(as long as it is negative which is not a requirement) may be interpreted as the heat dissipation rate $\int fj\,dx$. Let us consider the stationary regime $\dot{S}=0$ associated with an (*a priori* assumed to exist) invariant density ρ_*. Then,

$$b = u = D\,\nabla\,\ln\rho_*$$

and

$$-(1/k_BT)\,\nabla\,\mathcal{V} = \nabla\ln\rho_* \Rightarrow \rho_* = \frac{1}{Z}\exp[-\mathcal{V}/k_BT]. \tag{90}$$

Hence

$$-\gamma s_* = \mathcal{V} + k_BT\ln\rho_* \Rightarrow \Psi_* = -k_BT\ln Z \doteq \gamma F_* \tag{91}$$

with $Z=\int\exp(-\mathcal{V}/k_BT)dx$. Ψ_* stands for the minimum of the time-dependent Helmholtz free energy Ψ. Because

$$Z = \exp(-\Psi_*/k_BT), \tag{92}$$

we have

$$\rho_* = \exp[(\Psi_* - V)/k_BT]. \tag{93}$$

Therefore, the conditional Kullback-Leibler entropy $\mathcal{H}_c$ of the density ρ relative to an equilibrium (stationary) density ρ_* acquires the form

$$k_BT\mathcal{H}_c \doteq -k_BT\int\rho\ln\left(\frac{\rho}{\rho_*}\right)dx = \Psi_* - \Psi. \tag{94}$$

In view of the concavity property of the function $f(w)=-w\ln w$, $\mathcal{H}_c$ takes only negative values, with a maximum at 0. We have $\Psi_*\leqslant\Psi$ and $k_BT\dot{\mathcal{H}}_c=-\dot{\Psi}\geqslant 0$. $\mathcal{H}_c(t)$ is bound to grow monotonically toward 0, while $\Psi(t)$ drops down to its minimum Ψ_*, which is reached at ρ_*.

The Helmholtz free energy minimum remains divorced from any extremal property of the Gibbs-Shannon entropy. Only the Kullback-Leibler entropy shows the expected (growth) asymptotic behavior. See, e.g., also [2].

Note that the properties of the free Brownian motion can be easily inferred by setting $b\equiv 0$ in the above discussion. Then the diffusive dynamics is sweeping and there is no asymptotically invariant density, nor a finite minimum for $\Psi(t)$, which decreases indefinitely.

2. Reintroducing duality

To set a connection with the previous time duality (imaginary time transformation) framework, we need only to observe some classic properties of Smoluchowski diffusion processes. Once we set $b=-2D\nabla\Phi$ with $\Phi=\Phi(x)$, the substitution

$$\rho(x,t) \doteq \theta_*(x,t)\exp[-\Phi(x)] \tag{95}$$

with θ_* and Φ being real functions, converts the Fokker-Planck equation into a generalized diffusion equation for θ_*:

$$\partial_t\theta_* = D\Delta\theta_* - \frac{V(x)}{2mD}\theta_* \tag{96}$$

and its time adjoint (here trivialized in view of the time independence of Φ)

$$\partial_t\theta = -D\Delta\theta + \frac{V(x)}{2mD}\theta \tag{97}$$

for a real function $\theta(x,t)=\exp[-\Phi(x)]$, where

$$\frac{V(x)}{2mD} = \frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right) = D[(\nabla\Phi)^2 - \Delta\Phi]. \tag{98}$$

Let us note the obvious factorization property for the Fokker-Planck probability density

$$\rho(x,t) = \theta(x,t)\theta_*(x,t), \tag{99}$$

which stays in affinity with a quantum mechanical factorization formula $\rho=\psi^*\psi$, albeit presently realized in terms of two real functions θ and θ^*, instead of a complex conjugate pair. In view of (at this point we restore the original notation of Sec. IV)

$$\bar{\rho}^{1/2}(x,t)\exp[-\bar{s}(x,t)/2mD] \doteq \theta_*(x,t), \tag{100}$$

we immediately recover

$$\bar{s} = (2mD)[\Phi - (1/2)\ln\bar{\rho}], \tag{101}$$

in conformity with the previous definition Eq. (83). If there are no external forces, Φ disappears, and we are left with the free Brownian motion associated with $\bar{s}=-mD\ln\bar{\rho}$.

For the record it is useful to mention explicit transformations between Green's functions appropriate for quantum motion and transition probability densities of standard diffusion type processes [25]. Explicit examples of the free dual dynamics and those in the harmonic potential have been worked out there.

VI. CONCLUSIONS

An analytic continuation in time (or an imaginary time transformation) stands for a mapping between two different types of dynamics, both running with respect to the equally real(istic) time. In our case, the modular Schrödinger equation has been the unifying departure point for an analysis of (sometimes not quite expected) affinities between dynamical patterns of behavior generated by the same primary nonlinear field, but typically considered disjointly—as research problems on their own, for properly selected coupling parameter values.

One may possibly take the view that the $\kappa=0,1,2$ cases are just formally analogous mathematical descriptions of different physical systems. Our standpoint is that a fundamental dynamical system is a modular Schrödinger equation with an arbitrarily adjustable coupling parameter. Different coupling regimes refer to physically different patterns of behavior, but there is a deep connection between them, to be further explored. The global changes of properties of solutions of the nonlinear dynamical equations as the control parameter is varied are routinely known to nonlinear system experts. This property is shared by the modular Schrödinger equation as well.

If one takes seriously the dynamical duality (or time duality) concept, the models considered here should not longer be viewed independently. One can trace the dynamical patterns of one model in terms of those for another, and in reverse, even, if at first glance, the pertinent dynamics patterns may seem to have nothing in common.

It is the Hamiltonian analysis of the nonlinear field (i.e., the Schrödinger equation) which has led to a variety of emerging Hamiltonian motion scenarios. Two classes of them (we call them confining and scattering) stay in close affinity on the quantum, classical, and stochastic (dissipative) dynamics levels of description.

Euclidean methods are often used in various (especially quantum) branches of statistical physics research on equilibrium and near-equilibrium phenomena. Our discussion was basically concentrated on the real time flow notion, which, even after a Euclidean (imaginary time) transformation, still remains a real(istic) time flow, albeit with a new physical meaning.

It is worth mentioning that an an independent, quantum-theory-motivated approach [26] (deformation quantization with an imaginary transformed deformation constant $\hbar$) has obvious links with the Euclidean map viewpoint toward diffusion-type processes, explored in the present paper. We would like to point out that, quite aside from existing and prospective physical implementations, the sign-inversion issue for conservative potentials has a deeper mathematical meaning whose role we have slightly diminished, not to overburden the text with a strong admixture of advanced functional analysis. The dynamical semigroup indications in the present paper were intended to tell the mathematically interested reader under what circumstances one can be sure of the existence (modulo suitable time interval limitations) of solutions that are connected by an imaginary time transformation. The semigroup notion cannot be hastily extended to classical dynamics; nonetheless an imaginary time link still works there.

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